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Closed generalized Mazurkiewicz sets are curves

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Abstract

Mazurkiewicz proved the existence of a subset of the Euclidean plane E^2 with the property that every straight line intersects it in exactly two points. A set with this property is called a *Mazurkiewicz set*. A nondegenerate subset X of E^2 is a *generalized Mazurkiewicz set* if each line that separates two points of X intersects X in exactly two points. We prove that a generalized Mazurkiewicz set must be a simple closed curve if it contains an arc. From this we deduce that a closed, generalized Mazurkiewicz set is a simple closed curve. Simple closed curves in E^2 are generalized Mazurkiewicz sets if and only if they bound convex disks.

Keywords: Convex; Generalized Mazurkiewicz sets; Mazurkiewicz sets; Midsets; Planar sets; Simple closed curve; Straight lines; Two-point sets

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1. Introduction

Sylvester's theorem states that for each finite, noncollinear subset X of E^2 there must exist at least one *ordinary line* (see [2, p. 32]); that is, there must be at least one line that intersects X in exactly two points. At the other extreme, Mazurkiewicz [9] proved the existence of a subset of E^2 with respect to which every line is ordinary. A set is called a *Mazurkiewicz set* if every line intersects it exactly twice. Clearly no bounded set can be a Mazurkiewicz set, and Mauldin [8] proved that each Mazurkiewicz set is totally disconnected. However, generalized Mazurkiewicz sets can be bounded, connected, or totally disconnected because only those lines that separate two points of the set are required to be ordinary

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lines. For example circles, triangles, squares, and ellipses are all generalized Mazurkiewicz sets but so is every Mazurkiewicz set. By our definition, a generalized Mazurkiewicz set must contain at least two points. We seek minimal additional conditions on a generalized Mazurkiewicz set sufficient to ensure that it is a simple closed curve.

One of our two main theorems states that a generalized Mazurkiewicz set that contains an arc must be a simple closed curve (Theorem 2.6). From this we deduce that a closed, generalized Mazurkiewicz set X must be a simple closed curve (Theorem 2.12). Examples are easily produced to show the necessity of "closed" in the hypothesis. Larman [3, Theorem 2] showed that no Mazurkiewicz set can be closed, a fact that also follows from Theorem 2.12. Of course it follows from Theorem 2.6 that a path-connected generalized Mazurkiewicz set is also a simple closed curve.

From Lemma 2.3 it is clear that among simple closed curves in E^2, those that are generalized Mazurkiewicz sets are precisely those that bound convex disks.

A Mazurkiewicz set is also known in the literature as a *two-point set* [8], but a two-point set also designates a set consisting of two points. We use *Mazurkiewicz set* to avoid this confusion.

In Section 3 we briefly discuss a related two-point property (the double midset property) and generalizations to sets that each line intersects in exactly n points.

2. Definitions, lemmas, and theorems

A *continuum* is a nondegenerate (contains more than one point), compact, connected metric space, an *arc* is a homeomorphic image of the closed interval [0, 1] on the real line, and a *simple closed curve* is a homeomorphic image of the unit circle in E^2. Let $L(p, q)$ denote the straight line determined by the two points p and q, and let $R(p, q)$ denote the closed ray from p through q. The notation $[p, q]$ is used to denote the straight-line segment joining p and q, and (p, q) denotes $[p, q]-\{p, q\}$. A set X in E^2 is *convex* if the segment $[p, q]$ lies in X for every choice of p and q in X, and a simple closed curve in E^2 is called a *convex simple closed curve* if the disk it bounds is convex. The *closure* of a set X is denoted by $\overline{X}$, and $d(p, q)$ denotes the standard distance between p and q in E^2. If α is an arc, Int α designates the arc α with its endpoints removed.

Lemma 2.1. *If X is a generalized Mazurkiewicz set in E^2, α is an arc in X, a and b are the endpoints of α, L is a line such that $L \cap \alpha=\{a, b\}$, and p is a point of $X-\alpha$, then p and* Int α *cannot lie in the same component of E^2-L.*

Proof. Let S be the component of E^2-L containing Int α, and suppose $p \in S$. Among the lines parallel to L that intersect α, choose L' to be the one at the greatest distance from L, and let $t \in L' \cap \alpha$. Note that p cannot lie between L and

L' since X is a generalized Mazurkiewicz set, so p either lies on L' or L' separates p from L. If $p \in L'$, rotate L' slightly about p to obtain a line L'' that separates $\{a, b\}$ from t. Then L'' must intersect α in at least two distinct points. However, since $p \in (X-\alpha) \cap L''$, this contradicts the hypothesis on X. In the other case where $p \notin L'$, it follows because X is a generalized Mazurkiewicz set that $L' \cap X$ consists of two points $t \in \alpha$ and t'. Suppose $t' \in \alpha$. Then there is a line parallel to L' that intersects α in more than two points, contradicting the hypothesis on X. This shows $t' \notin \alpha$. Now rotate L' as before, this time about t', to obtain a line L^* that separates t from $\{a, b\}$. Then, since X is a generalized Mazurkiewicz set, L^* can intersect X only twice, yet L^* contains t' and must intersect α at least twice. This contradiction establishes the lemma. □

Lemma 2.2. *If X is a generalized Mazurkiewicz set in E^2, α is an arc in X, and L is a line through the endpoints of α, then there is a component S of $E^2 - L$ such that $S \cap \alpha = \emptyset$ and $X - \alpha \subset \overline{S}$.*

Proof. If $\alpha \subset L$, there must be a point p of X that does not lie in L since X is a generalized Mazurkiewicz set. Let S be the component of $E^2 - L$ that contains p, and let q be an arbitrary point in $X - \alpha$. If $q \notin \overline{S}$, then L separates p from q, which contradicts that X is a generalized Mazurkiewicz set. Thus $q \in \overline{S}$, and the conclusion of the lemma follows in this case.

If $\alpha \not\subset L$, choose a subarc α' of α such that L contains the endpoints a and b of α' and $L \cap (\text{Int } \alpha') = \emptyset$. Denote by S the component of $E^2 - L$ that does not intersect Int α. Then $X - \alpha \subset X - \alpha' \subset \overline{S}$ by Lemma 2.1. □

Lemma 2.3. *A simple closed curve J in E^2 bounds a convex disk if and only if J is a generalized Mazurkiewicz set.*

Proof. Suppose J is a generalized Mazurkiewicz set, and K is the disk bounded by J. Let x and y be two points of K, which may be assumed to lie in J, and let A and A' be the arcs whose union is J such that $A \cap A' = \{x, y\}$. Two applications of Lemma 2.2, one with $\alpha = A$ and the second with $\alpha = A'$, reveal that A and A' lie in the closures of opposite sides of the line $L(x, y)$. Then $[x, y] \subset K$, and K is convex.

The other direction is clear because a line through the interior of a convex disk must intersect its boundary in exactly two points. □

Lemma 2.4. *If a generalized Mazurkiewicz set X in E^2 contains a simple closed curve J, then $X = J$.*

Proof. Suppose there is a point $x \in X - J$, and let L be a line through x that also intersects the interior of J. Then L must intersect X in at least three points, which contradicts the hypothesis. □

Lemma 2.5. *Let X be a generalized Mazurkiewicz set, θ^* be an arc in X, and q, r, s be noncollinear points of X such that $q, r \in \operatorname{Int} \theta^*$. Let θ be the arc in θ^* with endpoints q, r, and assume θ does not intersect the s-side S of $L(q, r)$. If $L(q, s)$ and $L(r, s)$ each separate some two points of X, then*

(1) *for each $x \in (r, s]$, the ray $R(q, x)$ intersects $X - \{q\}$ in a single point $f(x)$,*
(2) *the function f is a homeomorphism of $(r, s]$ into $X - \{q\}$ such that $f(s) = s$,*
(3) *f extends to a homeomorphism of the interval $[r, s]$ such that $f(r) \in L(q, r)$,*
(4) *either $f(r) = r$ or the interval $[r, f(r)]$ lies in X, and*
(5) *it follows that there is an arc θ_1 in $(X - \{q\}) \cap \bar{S}$ having r and s as its endpoints.*

Proof. Using lines from s and the Mazurkiewicz set property, we see that X does not intersect the interior of the triangle $\triangle(q, r, s)$. By Lemma 2.2, $X - \theta \subset \bar{S}$ and $\theta \cap S = \emptyset$. For each $x \in (r, s]$, the line $L(q, x)$ separates some two points of X and must intersect $X - \{q\}$ in a single point $f(x)$ since X is a generalized Mazurkiewicz set. Suppose $f(x) \notin R(q, x)$. Then $f(x) \notin \bar{S}$. Since $X - \theta \subset \bar{S}$ we know $f(x) \in \theta$. Then θ contains an arc that intersects $L(q, x)$ only at its endpoints. But this contradicts Lemma 2.1 since $L(q, x)$ separates points of $X \cap S$. Thus $f(x) \in R(q, x)$, and $f(x) \notin \operatorname{Int} \triangle(q, r, s)$. This completes (1) in the itemized conclusions of the lemma.

Since $L(q, s)$ separates two points of X and X is a generalized Mazurkiewicz set, $L(q, s) \cap X = \{q, s\}$. This means $f(s) = s$. To show that f is continuous at $x \in (r, s]$, let $\{x_i\}$ be a sequence of points of (r, s) converging to x. We shall show $\{f(x_i)\}$ must converge to $f(x)$.

Suppose a subsequence of $\{x_i\}$ exists, say $\{x_i\}$ itself, such that $\lim_{i \to \infty} d(q, f(x_i)) = \infty$. Then for sufficiently large i, $L(f(x), f(x_i))$ intersects X in a third point in θ^*, which is a contradiction. It is also clear that no such subsequence can exist with $\{f(x_i)\}$ converging to q because $X \cap \operatorname{Int} \triangle(q, r, s) = \emptyset$. Suppose there exists a limit point p of the set $\{f(x_i)\}$ in $R(q, x)$ such that $p \neq f(x)$. Then for $f(x_i)$ sufficiently close to p, $L(f(x), f(x_i))$ intersects X in a third point in θ^*, contradicting the hypothesis. It follows that $\{f(x_i)\}$ must converge to $f(x)$, and f is continuous on the half open segment $(r, s]$. Let α denote the open arc $f((r, s])$ in X. This completes (2) since f is continuous and injective.

To extend f to $[r, s]$, let K denote the collection of all limit points of α in $R(q, r)$, and let $\{r_i\}$ be a sequence of points of (r, s) converging to r. Again, q is not in K because $X \cap \operatorname{Int} \triangle(q, r, s) = \emptyset$, and, for the same reason $K \cap (q, s) = \emptyset$. If $K = \emptyset$, then points x and y exist in α such that $L(x, y)$ separates r from s and intersects $L(q, s)$. But this contradicts Lemma 2.1 because the interior of the subarc of α bounded by $\{x, y\}$ can lie on neither side of $L(x, y)$. If K contained two points, then a line separating them would separate some two points of X and would intersect α in infinitely many points, contrary to the hypothesis. Then K consists of a single point p. Define $f(r) = p$, and note that $\{f(r_i)\}$ converges to $f(r)$. Thus, f is continuous on $[r, s]$. Since f is injective, this completes (3).

If $f(r) = r$, parts (4) and (5) of the lemma are finished by letting θ_1 be the closure of α. If $f(r) \neq r$, we show that the segment $[r, f(r)]$ lies in X, so that the lemma is finished by letting θ_1 be $f([r, s]) \cup [r, f(r)]$.

Assume $r \neq f(r)$. Since both $L(q, s)$ and $L(r, s)$ separate some two points of X, we may now interchange the roles of q and r and apply the first three parts of the conclusion of the lemma, which have already been proven, to obtain a continuous function $g:[q, s] \to \overline{S} \cap (X - \{r\})$ such that the arc $\beta = g([q, s])$ lies in X. Let $z \in (r, f(r)]$, and let L be a line through z that separates q from s. By hypothesis, $L \cap X$ consists of two points v and w, one of which, say v, lies in β. Then w cannot lie in β by Lemma 2.1. Also, using Lemma 2.1, one can show that α lies in the closure of the side of $L(f(r), s)$ opposite r, so $L \cap \alpha = \emptyset$, and it follows that $w \notin \alpha$. Suppose $z \neq w$. Using the generalized Mazurkiewicz property one sees that $w \notin S$. Since $X - \theta \subset \overline{S}$, $w \in \theta$. But this contradicts the two-point hypothesis on X because a line through z close enough to L to separate w from $\{q, r\}$ would have to intersect θ at two points and would also intersect β. Then $w = z \in X$, and, since z was arbitrary, $[r, f(r)] \subset X$ as desired. This completes the proof of the lemma. □

Theorem 2.6. *If a generalized Mazurkiewicz set X in E^2 contains an arc, then X is a simple closed curve.*

Proof. Let q^* and r^* be the endpoints of an arc θ^* in X, and let s and s' be the two points of $X \cap B$, where B is the perpendicular bisector of $[q^*, r^*]$. By Lemma 2.1, we may assume $s' \in \theta^*$ and $s \notin \theta^*$. Choose points q and r in the interior of θ^* such that each of the lines $L(q, s)$ and $L(r, s)$ separates q^* from r^*, and let θ be the arc in θ^* with endpoints q and r. Since $L(q, s)$ and $L(r, s)$ each separate points of X, it follows from the two-point hypothesis on X that q, r, and s cannot lie on the same line.

Since it follows from Lemma 2.2 that θ cannot intersect the s-side of $L(q, r)$, we may apply Lemma 2.5 to obtain an arc θ_1 in $X - \{q\}$ with endpoints r and s. From Lemma 2.2, θ_1 cannot intersect the q-side of $L(r, s)$. Apply Lemma 2.5 again, this time reversing the roles of q and r, to obtain an arc θ_2 in $X - \{r\}$ from s to q. Since θ_2 cannot intersect the r-side of $L(q, s)$, it follows that $\theta \cup \theta_1 \cup \theta_2$ is a simple closed curve. Lemma 2.4 completes the proof. □

Corollary 2.7. *A subspace X of E^2 is a convex simple closed curve if and only if X is a path-connected, generalized Mazurkiewicz set.*

Corollary 2.8. *If X is a generalized Mazurkiewicz set in E^2, then X is a simple closed curve or X contains no continuum.*

Proof. Suppose X contains a continuum M. It follows that M is locally connected, for otherwise some line would intersect M, and hence X, in infinitely many points (see [10, p. 90] or [11, 28D]). By the Hahn–Mazurkiewicz Theorem (see [11, Theorems 31.1 and 31.5]) M contains an arc, and by Theorem 2.6, X is a simple closed curve. □

Lemma 2.9. *If X is a closed set in E^2, $o \in X$, and every line through the point o intersects $X - \{o\}$ in exactly one point, then X contains an arc.*

Proof. Assume o is the origin, let C be the circle of radius 1 centered at o, and let $r: X - \{o\} \to C$ be the radial injection given by $r(\vec{x}) = \vec{x} / \|\vec{x}\|$. For each positive integer n, define the annulus A_n to be $\{\vec{x} \mid 1/n \leqslant \|\vec{x}\| \leqslant n\}$ and $X_n = A_n \cap X$. Let E_+^2 and E_-^2 be the closed upper and lower half planes, respectively, and, for each n, define $X_n^+ = E_+^2 \cap X_n$, $X_n^- = E_-^2 \cap X_n$, $C^+ = E_+^2 \cap C$, and $C^- = E_-^2 \cap C$. Then $X - \{o\} = \bigcup_{n=1}^{\infty}(X_n^+ \cup X_n^-)$, and X_n^+ and X_n^- are closed for each n.

Define the antipodal map $a: C^- \to C^+$ by $a(\vec{x}) = -\vec{x}$. Then, for each n, $r(X_n^+)$ and $a(r(X_n^-))$ are closed sets, and $C^+ = \bigcup_{n=1}^{\infty}[r(X_n^+) \cup a(r(X_n^-))]$. Since C^+ is homeomorphic to a closed interval, it follows from a Baire theorem [11, p. 186] that there exist an integer k and an arc A in C^+ such that either $A \subset r(X_k^+)$ or $A \subset a(r(X_k^-))$. The two cases being similar, we assume the former. Then define $f: A \to X_k^+$ by $f(\vec{x}) = r^{-1}(\vec{x})$. Since X_k^+ is closed, bounded, and $o \notin X_k^+$, f must be continuous. Then X_k^+ contains the arc $f(A)$ and so does X. □

Lemma 2.10. *If X is a closed, generalized Mazurkiewicz set in E^2, then X contains an arc.*

Proof. Choose points o and p of X such that $L(o, p)$ separates some two points of X, and impose a rectangular coordinate system with origin at o and with p on the positive x-axis. If each line through o intersects $X - \{o\}$ exactly once, which would be the case if each such line separates two points of X, then it follows from the previous lemma that X contains the desired arc. In the other case, there must exist a line N through o that fails to separate any two points of X. Since $L(o, p)$ separates two points of X, $N \neq L(o, p)$. This means there is a component S' of $E^2 - N$ that contains p, and X lies in $\bar{S}'$. Let S be the open sector at o whose closure contains X and whose central angle at o is minimal. Then $p \in S \subset S'$. Let C^* be the unit circle at o, and let $C = C^* \cap S$. Define the radial retraction $r: X - \{o\} \to \bar{C}$ as in the previous lemma, and note that r maps $(X - \{o\}) \cap S$ injectively onto C. For each integer n, define $C_n = \{c \in C \mid 1/n \leqslant \|r^{-1}(c)\| \leqslant n\}$. Since X is closed, C_n is closed in C for each n, and $C = \bigcup_{n=1}^{\infty} C_n$. Since C is homeomorphic to the real line, it follows from a Baire theorem [11, p. 186], that there exists an integer k and an arc A such that $A \subset C_k$. Define $f: A \to X$ by $f(a) = r^{-1}(a)$, for each $a \in A$. To see that f is continuous, assume $\{a_i\}$ is a sequence of points of A that converges to a point a of A. Then $\|f(a_i)\| \geqslant 1/k$ for each i, so o is not a limit point of $\{f(a_i)\}$. Also, since $\|f(a_i)\| \leqslant k$, for each i, the sequence is bounded. But X is closed and the ray $R(o, a)$ intersects X only at

$f(a)$, so $\{f(a_i)\}$ must converge to $f(a)$. Since f is a continuous injection on the compact set A, f is a homeomorphism, and $f(A)$ is an arc in X. □

The next corollary follows from Lemma 2.10 and Theorem 2.6. It is a special case of [3, Theorem 2].

Corollary 2.11. *No Mazurkiewicz set is closed.*

Theorem 2.12. *A subspace X of E^2 is a convex simple closed curve if and only if X is a closed, generalized Mazurkiewicz set.*

Proof. One direction of the proof follows from Lemma 2.3. In the other direction, to show that X is a simple closed curve, it suffices by Theorem 2.6, to show that X contains an arc. But this follows from the previous lemma, so the proof that X is a simple closed curve is complete. From Lemma 2.3, X bounds a convex disk. □

3. Remarks

The "Mazurkiewicz set" hypothesis in Theorems 2.6 and 2.12 cannot be weakened to just having all vertical and all horizontal lines intersect X in two points. Nor is it sufficient in these results to hypothesize the existence of a line L such that each line not parallel to L intersects X in exactly two points. We leave it to the reader to find simple examples of sets in E^2 with these properties where X is not a simple closed curve.

Define a set X in E^2 to have the *n-point intersection property* if each line intersects X in precisely n points, and define a nondegenerate set X to have the *generalized n-point intersection property* if every line that separates points of X intersects X in exactly n points. Thus, a generalized Mazurkiewicz set is a set with the generalized two-point intersection property. There are no sets with the one-point intersection property, but, for each $n > 1$, the existence of a set with the n-point intersection property has been established [1]. We do not know if there exist closed or connected, generalized n-point sets for $n > 2$; however,we suspect there are neither. It follows from [6] that there are no continua in E^2 with the generalized n-point intersection property ($n > 2$) because such a set would have the n-point midset property described and ruled out in [6].

All straight lines are required to be ordinary lines relative to a Mazurkiewicz set, but for a generalized Mazurkiewicz set only those lines that separate two points of the set are required to be ordinary. Not even all the separating lines are required to be ordinary in a set with the double midset property. A subset X of E^2 has the *double midset property* if all those lines that bisect some "chord" of X are ordinary. If a continuum in E^2 has this weaker two-point intersection property (the double midset property), then it must be a simple closed curve [5]. However, the result is not known when X is removed from E^2 [7]; in fact, a continuum X has been conjectured to be an n-sphere if each of its midsets (the set of all points of X equidistant from a given two points of X) is an $(n-1)$-sphere. Actually, a nondegenerate compact metric space X should be an n-sphere if each of its midsets is a topological $(n-1)$-sphere and $n \geqslant 2$. Perhaps one may also dispense with the "compactness" hypothesis on X. It is easy to prove X is connected when $n \geqslant 2$ and each midset of X is an $(n-1)$-sphere. See [4] for a more general theorem and related conjectures.

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